

Office Hours: 1:30-3:00 in COM B-006

7.1 Integration by Parts (continued)

Summary: $\int u dv = uv - \int v du$

1. Pick $u = ??$. The rest is dv .
2. Compute du and v .

Entry Task: Evaluate

$$1. \int \frac{\ln(x)}{\sqrt{x}} dx$$

$$u = \ln(x) \quad dv = x^{-1/2} dx$$
$$du = \frac{1}{x} dx \quad v = 2x^{1/2} = 2\sqrt{x}$$

$$= 2\sqrt{x} \ln(x) - \int \underbrace{2\sqrt{x}}_{x^{-1/2}} \frac{1}{x} dx$$

$$= 2\sqrt{x} \ln(x) - 2(2x^{1/2}) + C$$

$$= \boxed{2\sqrt{x} \ln(x) - 4\sqrt{x} + C}$$

CHECK

$$\frac{1}{\sqrt{x}} \ln(x) + 2\sqrt{x} \frac{1}{x} - \frac{2}{\sqrt{x}} \quad \checkmark$$

$$2. \int_0^1 x^2 e^{x/3} dx$$

$$u = x^2 \quad dv = e^{1/3 x} dx$$
$$du = 2x dx \quad v = 3e^{1/3 x}$$

$$= 3x^2 e^{1/3 x} \Big|_0^1 - \int_0^1 6x e^{1/3 x} dx$$

$$= 3e^{1/3} - \left[18x e^{1/3 x} \Big|_0^1 - \int_0^1 18e^{1/3 x} dx \right]$$

$$u = 6x \quad dv = e^{1/3 x} dx$$

$$du = 6 dx \quad v = 3e^{1/3 x}$$

$$= 3e^{1/3} - 18e^{1/3} + 18 \cdot 3 e^{1/3 x} \Big|_0^1$$

$$= -15e^{1/3} + 54(e^{1/3} - e^0)$$

$$= \boxed{39e^{1/3} - 54}$$

Integration by parts is good for:

Products: $\underbrace{x} \underbrace{e^x}, \underbrace{x^2} \cos(3x), \underbrace{x} \sin(5x)$

Logs: $\underbrace{\ln(x)}, \underbrace{x^{10}} \underbrace{\ln(x)}, \frac{\overbrace{\ln(x)}^{\leftarrow}}{x^3}, \dots$

Inv. Tri: $\underbrace{\sin^{-1}(x)}, \underbrace{x} \tan^{-1}(x), \dots$

Products: $\underbrace{e^x} \sin(x), \underbrace{e^x} \cos(x)$

Example:

$$\int \frac{\sin^{-1}(x)}{\tan^{-1}(x)} dx = \int \frac{\arcsin(x)}{\arctan(x)} dx$$

$$u = \tan^{-1}(x) \quad dv = dx \\ du = \frac{1}{1+x^2} dx \quad v = x$$

$$= x \tan^{-1}(x) - \int \frac{x}{1+x^2} dx$$

$$= x \tan^{-1}(x) - \int \frac{x}{u} \cdot \frac{1}{2x} du$$

$$u = 1+x^2 \\ du = 2x dx \\ \frac{1}{2x} du = dx$$

$$= x \tan^{-1}(x) - \frac{1}{2} \ln|u| + C$$

$$= \boxed{x \tan^{-1}(x) - \frac{1}{2} \ln(1+x^2) + C}$$

ASIDE:

$$y = \tan^{-1}(x) \\ \Rightarrow \tan(y) = x$$

$$\Rightarrow \sec^2(y) \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2(y)}$$

$$= \frac{1}{1+\tan^2(y)}$$

$$= \frac{1}{1+x^2}$$

Example: (Never ending integration by parts and how to end it):

$$\int e^x \cos(x) dx$$

$$u = e^x \quad dv = \cos(x) dx$$

$$du = e^x dx \quad v = \sin(x)$$

$$e^x \sin(x) - \int e^x \sin(x) dx$$

$$u = e^x \quad dv = \sin(x) dx$$

$$du = e^x dx \quad v = -\cos(x)$$

$$e^x \sin(x) - (-e^x \cos(x) - \int e^x \cos(x) dx)$$

$$\Rightarrow \int e^x \cos(x) dx = e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x) dx$$

$$2 \int e^x \cos(x) dx = e^x \sin(x) + e^x \cos(x) + C_0$$

$$\int e^x \cos(x) dx = \frac{1}{2} e^x \sin(x) + \frac{1}{2} e^x \cos(x) + \frac{C_0}{2}$$

7.2 Trigonometric Integral Methods

Goal: A procedure to integrate *any* combination of trig functions.

Motivating examples: These are substitution problems, what is u?

$$\int \sin^3(x) (1 - \sin^2(x)) \frac{\cos(x) dx}{du}$$

$$\int (1 - \cos^2(x)) \cos^5(x) \frac{\sin(x) dx}{-du}$$

$$\int \tan^5(x) (1 + \tan^2(x)) \frac{\sec^2(x) dx}{du}$$

$$\int \sec^6(x) \frac{\sec(x)\tan(x) dx}{du}$$

7.2 idea: Use trig identities to turn almost all trig problems into one of these situations!

Essential Tools

$$\tan(x) = \frac{\sin(x)}{\cos(x)}, \cot(x) = \frac{\cos(x)}{\sin(x)},$$
$$\sec(x) = \frac{1}{\cos(x)}, \csc(x) = \frac{1}{\sin(x)}.$$

See my online postings (or the Appendix of your book) for a more general discussion and proofs of trig identities.

$$\sin^2(x) + \cos^2(x) = 1$$

$$\tan^2(x) + 1 = \sec^2(x)$$

$$\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$$

$$\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$$

$$\sin(x) \cos(x) = \frac{1}{2} \sin(2x)$$

Case 1 (cosine or sine has an odd power)

i) $\int \sin^2(x) \cos^3(x) dx$ \leftarrow ODD
 PULL OUT ONE COSINE

$\int \sin^2(x) \cos^2(x) \cos(x) dx$
 $\int \sin^2(x) (1 - \sin^2(x)) \cos(x) dx$ \leftarrow USE IDENTITY

$\int u^2(1-u^2) du$ $u = \sin(x)$
 $\int u^2 - u^4 du = \frac{1}{3}u^3 - \frac{1}{5}u^5 + C = \frac{1}{3}\sin^3(x) - \frac{1}{5}\sin^5(x) + C$
 $du = \cos(x) dx$

ii) $\int \sin^3(x) dx$ \leftarrow ODD
 PULL OUT ONE SINE

$\int \sin^2(x) \sin(x) dx$ \leftarrow USE IDENTITY
 $\int (1 - \cos^2(x)) \sin(x) dx$

$\int (1 - u^2) \frac{1}{-u} du$ $u = \cos(x)$
 $du = -\sin(x) dx$
 $-\frac{1}{\sin(x)} du = dx$

$-\int 1 - u^2 du$

$-(u - \frac{1}{3}u^3) + C$

$-\cos(x) + \frac{1}{3}\cos^3(x) + C$

Case 2 (Both have even powers)

i) $\int \cos^2(x) dx$ \leftarrow even

$\int \frac{1}{2}(1 + \cos(2x)) dx$

$\frac{1}{2} \int 1 + \cos(2x) dx$

$\frac{1}{2}(\theta + \frac{1}{2}\sin(2x)) + C$

$\frac{1}{2}\theta + \frac{1}{4}\sin(2x) + C$

ii) $\int \sin^4(x) dx$

$\int \sin^2(x) \sin^2(x) dx$

$= \int \frac{1}{2}(1 - \cos(2\theta)) \frac{1}{2}(1 - \cos(2\theta)) d\theta$

$= \frac{1}{4} \int 1 - 2\cos(2\theta) + \cos^2(2\theta) d\theta$

$= \frac{1}{4} \left[\theta - \sin(2\theta) + \int \frac{1}{2}(1 + \cos(4\theta)) d\theta \right]$

$= \frac{1}{4}\theta - \frac{1}{4}\sin(2\theta) + \frac{1}{8}(\theta + \frac{1}{4}\sin(4\theta)) + C$

$= \frac{3}{8}\theta - \frac{1}{4}\sin(2\theta) + \frac{1}{32}\sin(4\theta) + C$

Case 3 (even power on secant)

$$\int \tan^2(x) \sec^4(x) dx \quad \left. \begin{array}{l} \leftarrow \text{even} \\ \text{PULL} \\ \text{OUT} \\ \sec^2(x) \end{array} \right\}$$

$$\int \tan^2(x) \sec^2(x) \sec^2(x) dx$$

$$\int \tan^2(x) (1 + \tan^2(x)) \sec^2(x) dx$$

$$u = \tan(x) \\ du = \sec^2(x) dx$$

$$\int u^2 (1 + u^2) du$$

$$\int u^2 + u^4 du$$

$$\frac{1}{3} u^3 + \frac{1}{5} u^5 + C$$

$$= \boxed{\frac{1}{3} \tan^3(\theta) + \frac{1}{5} \tan^5(\theta) + C}$$

Case 4 (Odd power on ~~secant~~^{tangent}, and at least one ~~tangent~~^{sec(x)})

$$\int \tan^3(x) \sec^5(x) dx \quad \left. \begin{array}{l} \leftarrow \text{ODD} \\ \text{PULL OUT} \\ \sec(x) \tan(x) \end{array} \right\}$$

$$\int \tan^2(x) \sec^4(x) \sec(x) \tan(x) dx$$

$$\int (\sec^2(x) - 1) \sec^4(x) \sec(x) \tan(x) dx$$

$$u = \sec(x) \\ du = \sec(x) \tan(x) dx$$

$$\int (u^2 - 1) u^4 du$$

$$\int u^6 - u^4 du$$

$$\frac{1}{7} u^7 - \frac{1}{5} u^5 + C$$

$$\boxed{\frac{1}{7} \sec^7(x) - \frac{1}{5} \sec^5(x) + C}$$

Notes:

And if you've tried all methods and are stuck, here are things to try:

1. Rewrite in terms of $\sin(x)$ and $\cos(x)$.
2. Rewrite in terms of $\sec(x)$ and $\tan(x)$.
3. Try using trig identities.

And there are still a few "holes".

Particularly, odd power on $\sec(x)$.

For these you can quote

(proof in the book):

$$\int \tan(x) dx = \ln|\sec(x)| + C$$

$$\int \sec(x) dx = \ln|\sec(x) + \tan(x)| + C$$

$$\int \sec^3(x) dx = \frac{1}{2} \sec(x) \tan(x) + \frac{1}{2} \ln|\sec(x) + \tan(x)| + C$$

What is the first step in each integral below?

$$\int \overset{\text{ODD}}{\sin^3(x)} \cos^4(x) dx = \int (1 - \cos^2(x)) \cos^4(x) \sin(x) dx$$

$$\int \overset{\text{ODD}}{\sin^5(x)} \overset{\text{ODD}}{\cos^3(x)} dx = \int \sin^5(x) (1 - \sin^2(x)) \cos(x) dx$$

$$\int \overset{\text{EVEN}}{\cos^4(x)} dx = \int \cos^2(x) \cos^2(x) dx = \int \frac{1}{2}(1 + \cos(2x)) \frac{1}{2}(1 + \cos(2x)) dx$$

$$\int \tan^5(x) \overset{\text{EVEN}}{\sec^4(x)} dx = \int \tan^5(x) (1 + \tan^2(x)) \sec^2(x) dx$$

$$\int \overset{\text{ODD}}{\tan^5(x)} \sec(x) dx = \int (\sec^2(x) - 1)^2 \sec(x) \tan(x) dx$$